Lecture Note 3: Irreducibility Criteria, Frobenius–Schur Analysis, and Modular Representation Implications

Overview:
In the third lecture of the series, we complete our analysis of A_5 over fields of positive characteristic. We finalize the discussion on irreducibility of representations derived from the decomposition matrix, apply the Frobenius–Schur theorem to determine the nature of the representations, and tie all results back to both classical and computational perspectives.

1. Finalizing Irreducibility of Modular Representations

Previously, we deduced a two-dimensional irreducible modular representation of A_5 over $\overline{\mathbb{F}}_2$, based on decomposition matrix arguments and trace computations. The last representation in question is verified by:

- Using GAP to compute the full decomposition matrix.
- Observing that any proper subrepresentation would imply the existence of a one-dimensional modular representation.
- Contradiction: A_5 is a non-abelian simple group and cannot have any non-trivial one-dimensional irreducible representations.

Conclusion: The remaining candidate must be irreducible.

2. Summary: Modular Representations of A_5 over $\overline{\mathbb{F}}_2$

Let mod $p = 2$. Then:

- There exist exactly three irreducible Brauer characters.
- The degrees of these characters are: $1, 2, 3$
- Total dimension-squared check:

$$1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14 < 60$$

(Expected: total is less because we are only summing over p-regular classes)

3. Applying Frobenius–Schur Theory

We now apply Frobenius–Schur indicators to classify representation types:

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

Interpretation:

- $\nu = 1$: real (orthogonal type)

- $\nu = -1$: quaternionic (symplectic type)
- $\nu = 0$: complex (not real)

This analysis relates to the underlying division ring structure in the Wedderburn decomposition of the group algebra.

4. Frobenius Automorphism and Conjugate Representations

Let $\mathbb{F}_q$ be a finite field. The Frobenius automorphism $\varphi : x \mapsto x^q$ acts on trace values:

- Given a modular representation with trace values in $\mathbb{F}_q$, applying Frobenius yields another representation.
- Often, these are non-isomorphic but Galois-conjugate representations.

In our case:

- From a representation over $\mathbb{F}_4$, we apply $x \mapsto x^2$ to obtain its Frobenius conjugate.
- This produces a pair of irreducible modular representations, as seen in the decomposition matrix.

5. GAP Perspective and Character Table Analysis

Computational algebra systems like GAP are essential in modern representation theory. They:

- Compute character tables and decomposition matrices
- Check orthogonality and nullspace relations
- Verify irreducibility via matrix techniques
- Handle field extensions and large groups

For A_5, one can compute:

```
tbl := CharacterTable("A5");
DecompositionMatrix(tbl, 2);
```

This confirms:

- Number of irreducible Brauer characters
- Their degrees
- Relations to ordinary characters

6. Higher-Level Theoretical Implications

These methods generalize to any finite group G:

- For simple groups, modular representations reveal minimal degrees and structural properties.
- Frobenius–Schur indicators classify real vs quaternionic representations.
- The decomposition matrix encodes reduction behavior in a compact form.

Corollary: For a finite simple group G, any non-trivial modular representation must be of dimension ≥ 2, and if irreducible, it cannot arise from a reduction of a 1-dimensional character.

7. Concluding Remarks

This series demonstrated:

- Modular representation theory from algebraic and computational perspectives
- Full analysis of A_5's characters over $\mathbb{F}_2$
- Importance of decomposition matrices and Frobenius theory

Key Takeaway: Character tables, though static, contain dynamic modular and structural information when interpreted through the right theoretical lens.